

Limit Groups are Subgroup Conjugacy Separable

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May 17, 2016

Abstract

A group G is called subgroup conjugacy separable if for every pair of non-conjugate finitely generated subgroups of G , there exists a finite quotient of G where the images of these subgroups are not conjugate. We prove that limit groups are subgroup conjugacy separable. We also prove this property for one relator groups of the form $R = \langle a_1, \dots, a_n \mid W^n \rangle$ with $n > |W|$. The property is also proved for virtual retracts (equivalently for quasiconvex subgroups) of hyperbolic virtually special groups.

1 Introduction

O. Bogopolski and F. Grunewald [6] recently introduced the important notion of subgroup conjugacy separability for a group G . A group G is said to be subgroup conjugacy separable if for every pair of non-conjugate finitely generated subgroups H and K of G , there exists a finite quotient of G where the images of these subgroups are not conjugate. They proved that free groups and the fundamental groups of finite trees of finite groups subject to a certain normalizer condition, are subgroup conjugacy separable. For finitely generated virtually free groups the result was proved in [8]. Also, O. Bogopolski and K-U. Bux in [5] proved that surface groups are conjugacy subgroup separable.

Surface groups belong to the class of limit groups, the object of extensive study in the last few decades due to the fact that they play a key role in the solution of the Tarski problem.

Our main result generalizes the result of Bogopolski and Bux.

* Both authors were supported by CNPq.

Theorem 1.1. *Let G be a limit group. Then G is subgroup conjugacy separable.*

Bogopolski and Grunewald in their paper used also a notion of into conjugacy separability. A subgroup H of a group G is called into conjugacy separable if for every finitely generated subgroup K not conjugate into H there exists a finite quotient of G where the image of K is not conjugate into the image of H . In this paper we do not need to ask for K to be finitely generated. So changing slightly the definition of Bogopolski-Grunewald we say that a subgroup H of a group G is into conjugacy distinguished if for subgroup K not conjugate into H there exists a finite quotient of G where the image of K is not conjugate into the image of H . In terms of the profinite completion it reads as follows: H is into conjugacy distinguished if every subgroup K of G , the closure \overline{K} is conjugate into \overline{H} in \widehat{G} if and only if K is conjugate into H in G . We show in the paper that every finitely generated subgroup of a limit group is into conjugacy distinguished.

The methods of the proof are based on the paper [17] of Ribes and the second author on groups whose finitely generated subgroups are conjugacy distinguished. In particular, we use ideas of Section 3 from that paper, where the virtual retract property plays a crucial role.

This allows to extend Theorem 1.1 to virtual retracts of hyperbolic groups with conjugacy separable finite index subgroups.

Theorem 1.2. *Let G be a hyperbolic group such that every finite index subgroup of G is conjugacy separable and let H be a virtual retract of G . Then H is into conjugacy distinguished. In particular G is a virtual retracts subgroup conjugacy separable.*

A group G is called virtually special if there exists a special compact cube complex X having a finite index subgroup of G as its fundamental group (see [21] for definition of special cube complex). Virtually special groups owe its importance to Daniel Wise who proved in [21] that 1-relator groups with torsion are virtually special, answering positively a question of Gilbert Baumslag who asked in [3] whether this groups are residually finite. In fact, many groups of geometric origin are virtually special: the fundamental group of a hyperbolic 3-manifold (Agol [1]), small cancellation groups (a combination of [21] and [1]) and hyperbolic Coxeter groups (Haglund and Wise [10]) are virtually special.

Moreover, Haglund and Wise in [11] showed that quasiconvex subgroups of a virtually special hyperbolic group G (i.e., a subgroups that represents a quasiconvex subset in the set of vertices of the Cayley graph of G) are virtual retracts of G . Thus the next theorem applies in particular to this important class of subgroups.

Theorem 1.3. *Let G be a hyperbolic virtually special group and let H be a quasiconvex subgroup of G . Then H is into conjugacy distinguished. In particular G is quasiconvex subgroup conjugacy separable.*

As an application of it we obtain

Theorem 1.4. *Let $R = \langle a_1, \dots, a_n \mid W^n \rangle$ be a one relator group with $n > |W|$. Then every finitely generated subgroup H of R is into conjugacy distinguished and R is subgroup conjugacy separable.*

After this paper was submitted Bogopolski and Bux put the paper [5] into arxiv, where they gave independent prove of Theorems 1.2 and 1.3 in [5, Lemma 6.2 and Corollary C] for torsion free groups. Our methods allow us to avoid the assumption of torsion freeness. In particular, the case of small cancelation groups groups with finite $C'(1/6)$ or $C'(1/4) - T(4)$ presentations is covered by our results.

We finish the paper showing that a direct product of two free groups is not subgroup conjugacy separable (see Section 3).

Proposition 1.5. *A direct product $F_2 \times F_2$ of two free groups of rank 2 is not subgroup conjugacy separable.*

Acknowledgements: We thank Ashot Minasyan for conversations on the subject.

2 Proofs

A subgroup H of a group G is called a virtual retract if H is a semidirect factor (retract) of some finite index subgroup of G . A group G is called hereditarily conjugacy separable if every finite index subgroup of G is conjugacy separable.

Lemma 2.1. *Let G be a group, U a finite index conjugacy separable subgroup of G and H is a retract of U . Let K be a subgroup of G such that $\overline{K}^{\gamma} \leq \overline{H}$ for some $\gamma \in \widehat{G}$ and k be an element of K . Then there are G -conjugates H' and K' of H and K respectively, such that $(K')^{\gamma'} \leq \overline{H}'$ for some $\gamma' \in C_{\overline{U}}(k)$. Moreover, if $C_G(k)$ is virtually cyclic and U is hereditarily conjugacy separable, then the γ' can be achieved to be in $\langle k \rangle$.*

Proof. We shall replace H, K by their conjugates in G and change γ correspondingly until we achieve the statement of the lemma holding for them; thus the final H and K will be our H' and L' of the statement.

Note first that G is residually finite, since U is, so we can regard G as a dense subgroup of \widehat{G} . Since $G\widehat{U} = \widehat{G}$, replacing K by some conjugate in G we may assume that γ belongs to \widehat{U} and K is contained in U , since $\overline{H} \leq \overline{U}$ and U is closed in the profinite topology. By Proposition 7 in [17] H is conjugacy distinguished, therefore k is conjugated to an element of H in U . Hence, we may assume that k belongs to H .

Let $f : U \rightarrow H$ be the epimorphism with the restriction to H being the identity map and $\hat{f} : \widehat{U} \rightarrow \overline{H}$ be the continuous extension of it.

We have, $k^\gamma \in \overline{H}$, so $k^\gamma = \hat{f}(k^\gamma) = f(k)^{\hat{f}(\gamma)} = k^{\hat{f}(\gamma)} \in \hat{f}(\overline{H}) = \overline{H}$. Hence, $\hat{f}(\gamma)^{-1}\gamma \in C_{\widehat{G}}(k)$.

Replacing γ by $\hat{f}(\gamma)^{-1}\gamma$ we achieve that γ centralizes k .

Note however that if $C_G(k)$ is virtually cyclic, the group generated by k has finite index in $C_U(k)$, and since U is hereditarily conjugacy separable by Proposition 3.2 in [15] $C_U(k)$ is dense in $C_{\widehat{U}}(k)$. Hence,

$$\langle \widehat{h} \rangle C_U(k) = C_{\widehat{U}}(k).$$

It means that conjugating K by an element of $C_U(k)$, we may assume that $\gamma \in \langle \widehat{k} \rangle$. □

Our main tool is the following proposition whose proof uses essentially Proposition 7 in [17].

Proposition 2.2. *Let G be a hereditarily conjugacy separable group and H be a virtual retract of G . Let K be a subgroup of G having an element h such that $C_G(h)$ is virtually cyclic. Then \overline{K} is conjugate into \overline{H} in \widehat{G} if and only if K is conjugate into H in G . Moreover, if K is closed then \overline{K} is conjugate to \overline{H} in \widehat{G} if and only if K is conjugate to H in G .*

Proof. Suppose $\overline{K}^\gamma \leq \overline{H}$, where $\gamma \in \widehat{G}$.

By hypothesis H is a virtual retract of G . So there exist a finite index subgroup U of G such that H is a retract of U . Then by Lemma 2.1 we may assume that $\gamma \in \langle \widehat{h} \rangle$. This implies that $K \leq \overline{H}$, and since by Corollary 3.1.6 (b) [18] H is closed (i.e. $H = \overline{H} \cap G$) we have $K \leq H$.

Assuming in addition that $\overline{K}^\gamma = \overline{H}$ and K is closed we have $H = \overline{H} \cap G = \overline{K} \cap G = K$ that shows the last statement of the proposition. □

Corollary 2.3. *Let G be a hereditarily conjugacy separable group and H a finitely generated subgroup of G . Let U be a finite index subgroup of G such that $H \cap U$ is a retract of U . Let K be a subgroup of G having an element h such that $C_G(h^n)$ is virtually cyclic for every natural n . Then \overline{K} is conjugate into \overline{H} in \widehat{G} if and only if K is conjugate into H in G . Moreover, if K is closed then \overline{K} is conjugate to \overline{H} in \widehat{G} if and only if K is conjugate to H in G .*

Proof. Suppose $\overline{K}^\gamma \leq \overline{H}$, where $\gamma \in \widehat{G}$. By Proposition 2.2 $K \cap U$ is conjugate into $H \cap U$ in U so we may assume that $K \cap U \leq H \cap U$. Choose natural n such that $h^n \in K \cap U$. By Lemma 2.1 we may assume that $\gamma \in \langle \widehat{h^n} \rangle$. This implies that $K \leq \overline{H}$, and since $H = \overline{H} \cap G$ (indeed, $H \cap U$ is a retract of U , hence is closed and so H is closed), we have $K \leq H$.

Assuming in addition that $\overline{K}^\gamma = \overline{H}$ and K is closed we have $H = \overline{H} \cap G = \overline{K} \cap G = K$ that shows the last statement of the corollary. \square

To apply Proposition 2.2 to limit groups we shall need the following easy

Lemma 2.4. *Let G be a finitely generated non-abelian limit group. Then G has an element whose centralizer is cyclic.*

Proof. Let $G_n = G_{n-1} *_C A$ be n -th extension of centralizers (A is free abelian of rank m) such that $G \leq G_n$. Let $a, b \in G$ be non-commuting elements of G . Since G_n is commutative transitive, the centralizer of any element of G_n is free abelian and if it is non-cyclic it must intersect a conjugate of C by Theorem 14 [19] and so must be conjugate to A . We need to find an element in G not conjugate to an element of A . Therefore we may assume that $a \in A^g, b \in A^h, A^g \neq A^h$ for some $g, h \in G_n$ and in fact conjugating G by g^{-1} we may assume that $a \in A$. It follows then from the canonical normal form of ab in G_n that it can not be conjugate to an element of A in G_n . \square

Theorem 2.5. *Let G be a limit group. Then G is subgroup conjugacy separable. Moreover, every finitely generated subgroup of G is into conjugacy separable.*

Proof. Note first that G is hereditarily conjugacy separable (see Proposition 3.8 in [7]). Let H be a finitely generated subgroup of G . By Theorem B [20] H is a virtual retract of G . Let K be a finitely generated subgroup such that $\overline{K}^\gamma \leq \overline{H}$ for some $\gamma \in \widehat{G}$.

We distinguish two cases.

1. K is not abelian. Then by Lemma 2.4 there exists an element $k \in K$ whose centralizer is cyclic and the result follows from Proposition 2.2.

2. K is abelian. Let $k \neq 1$ be an element of K . By Lemma 2.1 we may assume that $K^\gamma \in \overline{H}$ for some $\gamma \in C_{\overline{G}}(k)$ and since $K \leq C_G(k) \leq C_{\widehat{G}}(k)$ by commutative transitivity property we have $K^\gamma = K$ and so $K \leq \overline{H} \cap G = H$ since H is closed in G . If $\overline{K}^\gamma = \overline{H}$ then the last formula gives the equality $K = H$. \square

Next we apply Corollary 2.3 to important groups of geometric nature.

Theorem 2.6. *Let G be a hereditarily conjugacy separable hyperbolic group and let H be a virtual retract of G . Then H is into conjugacy distinguished. In particular G is a virtual retract subgroup conjugacy separable.*

Proof. Observing that the centralizers of elements of infinite order of G are virtually cyclic (Proposition 3.5 [2]) one deduces the result from Corollary 2.3. \square

Proof of Theorem 1.3. By Theorem 1.1 in [16] G is hereditarily conjugacy separable and by [11] quasiconvex subgroups of G are virtual retracts. So the result follows from Theorem 2.6.

Theorem 2.7. *Let $R = \langle a_1, \dots, a_n \mid W^n \rangle$ be a one relator group with $n > |W|$. Then every finitely generated subgroup H of R is into conjugacy distinguished and R is subgroup conjugacy separable.*

Proof. By Theorem 1.4 in [21] R is hyperbolic virtually special and so every quasiconvex subgroup of it is a virtual retract by Proposition 4.3 in [4]. On the other hand by Theorem 1.2 in [13] every finitely generated subgroup of R is quasiconvex. Thus one deduces the result from the previous theorem. \square

Remark 2.8. *Let \mathcal{C} be a class of finite groups closed for subgroups, quotients and extensions. One can define then subgroup \mathcal{C} -conjugacy separability and prove the pro- \mathcal{C} version of Proposition 2.2 using Proposition 7 in [17] and Theorem 4.2 in [9] instead of Proposition 3.2 in [15].*

3 Direct product

We show here that a direct product of free groups of rank 2 is not subgroup conjugacy separable. It is based on an idea of Michailova combined with observations of V. Metaftsis and E. Raptis [14].

Consider a finitely presented group H given by G. Higman [12]

$$H = \langle x_1, x_2, x_3, x_4 \mid r_1, r_2, r_3, r_4 \rangle,$$

where $r_1 = x_2^{-1}x_1x_2x_1^{-1}$, $r_2 = x_3^{-1}x_2x_3x_2^{-1}$, $r_3 = x_4^{-1}x_3x_4x_3^{-1}$, $r_4 = x_1^{-1}x_4x_1x_4^{-1}$ and let F_4 be the free group on four generators x_1, x_2, x_3, x_4 .

Clearly, $F_4 \times F'_4$ can be considered as a finite index subgroup of $F_2 \times F'_2$, where F_2 and F'_4 are isomorphic copies of F_2 and F'_2 respectively. Since the induced profinite topology on a finite index subgroup is the full profinite topology, a subgroup of $F_4 \times F'_4$ is closed in the profinite topology of $F_4 \times F_4$ if and only if it is closed in the profinite topology of $F_2 \times F'_2$.

Let L_H be the subgroup of $F_4 \times F_4$ generated by

$$L_H = \langle (x_i, x_i), (1, r_i), i = 1, 2, 3, 4 \rangle.$$

Then $L_H \cap (F_4 \times \{1\})$ is the normal closure of $\langle r_1, r_2, r_3, r_4 \rangle$ in $(F_4 \times \{1\})$. So, by Proposition 1 in [14] L_H is closed in the profinite topology of $F_4 \times F'_4$, if and only if $L_H \cap (F_4 \times \{1\})$ is closed in the profinite topology of $(F_4 \times \{1\})$ or equivalently if and only if the group $H = \langle x_1, x_2, x_3, x_4 \mid r_1, r_2, r_3, r_4 \rangle$ is residually finite. However, G. Higman in [12] proved that H possesses no proper normal subgroups of finite index, so L_H is not closed in the profinite topology of $F_4 \times F'_4$.

Consider the closure $\overline{L_H}$ of L_H in $F_4 \times F'_4$. Then $\overline{L_H} \cap (F_4 \times \{1\})$ is a normal subgroup of $F_4 \times \{1\}$, and $(F_4 \times \{1\})/(\overline{L_H} \cap (F_4 \times \{1\}))$ is residually finite. On the other hand, $(F_4 \times \{1\})/(\overline{L_H} \cap (F_4 \times \{1\}))$ is a quotient of H , and H does not have any finite index normal subgroup, so $F_4 \times \{1\} = \overline{L_H} \cap (F_4 \times \{1\})$, consequently $\overline{L_H} = F_4 \times F'_4$. It means that L_H is a dense subgroup of $F_4 \times F'_4$.

Now, we show that $F_4 \times F'_4$ is not subgroup conjugacy separable. Indeed, since L_H is dense in $F_4 \times F'_4$, its image in each finite quotient coincides with the whole quotient. If L_H is not isomorphic to $F_4 \times F'_4$, then is not conjugate. Thus it suffices to prove that $L_H \not\cong F_4 \times F'_4$.

The subgroup $L_H \cap (F_4 \times \{1\})$ is an infinitely generated subgroup of $F_4 \times F'_4$ (since it is normal of infinite index), and it is clear that the centralizer of every element in $F_4 \times F'_4$ is finitely generated. Let $(e, 1)$ be a nontrivial element of $L_H \cap (F_4 \times \{1\})$. Then

$$C_{L_H}(e, 1) = C_{L_H \cap (F_4 \times \{1\})}(e) \times (L_H \cap (F_4 \times \{1\})),$$

so the right factor of the centralizer is infinitely generated and hence $C_{L_H}(e, 1)$ is infinitely generated as well. Therefore L_H and $\overline{L_H} = F_4 \times F'_4$ are not isomorphic subgroups.

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